Journal of Combinatorial Theory, Series B 99 (2009) 512–530

Contents lists available at ScienceDirect

Journal of Combinatorial Theory, Series B

www.elsevier.com/locate/jctb

A polynomial-time algorithm to find a linkless embedding of a graph ☆

Hein van der Holst

Department of Mathematics and Computer Science, Eindhoven University of Technology, PO Box 513, 5600 MB Eindhoven, The Netherlands

A R T I C L E   I N F O

Article history:
Received 28 June 2004
Available online 6 November 2008

Keywords:
Linkless embedding
3-space
Graphs
Algorithm
Polynomial-time

A B S T R A C T

A $\mathbb{Z}$-linkless embedding of a graph is an embedding in 3-space such that each pair of disjoint circuits has zero linking number. In this paper we present polynomial-time algorithms to compute a $\mathbb{Z}$-linkless embedding of a graph provided the graph has one and to test whether an embedding of a graph is $\mathbb{Z}$-linkless or not.

1. Introduction

For any two disjoint oriented closed curves C and D in 3-space and a diagram of them in some plane, the linking number $\mathrm{lk}(C,D)$ of C and D is the number of times C goes over D seeing D going from right to left minus the number of times C goes over D seeing D going from left to right, as in Fig. 1. The linking number is independent of the chosen diagram, and $\mathrm{lk}(C,D) = \mathrm{lk}(D,C)$. A *$\mathbb{Z}$-linkless* embedding of a graph G is an embedding of G in 3-space such that each pair of disjoint oriented circuits C, D has $\mathrm{lk}(C,D) = 0$.

In this paper we present a polynomial-time algorithm to compute a $\mathbb{Z}$-linkless embedding of a graph provided the graph has one. Moreover, if we are given an embedding of a graph, we can decide in polynomial time whether this embedding is a $\mathbb{Z}$-linkless embedding. As representation of embeddings of graphs we use diagrams of embeddings in some plane; that is, we use plane graphs in which some nodes are labelled as vertices and some nodes are labelled as undercrossing or overcrossing.

☆ Research for this work was done while the author was member of the European graduate program 'Combinatorics, Geometry, and Computation.' The European graduate program 'Combinatorics, Geometry, and Computation' is supported by the Deutsche Forschungsgemeinschaft, grant GRK 588/1.

E-mail address: h.v.d.holst@tue.nl.

0095-8956/$ – see front matter
doi:10.1016/j.jctb.2008.10.002

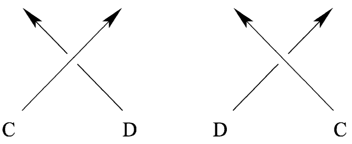

Fig. 1. The two types of crossings with C going over D.

A $\mathbb{Z}_2$-*linkless* embedding of a graph G is an embedding of G in 3-space such that each pair of disjoint circuits C, D has $\mathrm{lk}(C, D) = 0 \bmod 2$. A *linkless* embedding of G is an embedding of G in 3-space such that for each pair of disjoint circuits C, D, there is a topological hyperplane separating C from D. An embedding of G in 3-space is *flat* if each circuit is the boundary of a disc disjoint from G. To check whether a graph has a $\mathbb{Z}$-linkless embedding can be done with the results of Robertson, Seymour, and Thomas [4]. They showed that the following are equivalent for any graph G:

(i) G has a flat embedding;
(ii) G has a linkless embedding;
(iii) G has a $\mathbb{Z}$-linkless embedding;
(iv) G has a $\mathbb{Z}_2$-linkless embedding;
(v) G has no minor in the Petersen family.

The Petersen family is the family of all graphs that can be obtained from K_6, the complete graph on six vertices, by applying ΔY-transformations and $Y\Delta$-transformations. One graph in this family of graphs is the Petersen graph. Since testing whether a graph has a minor in a family of graphs can be done in polynomial time according to a theorem of Robertson and Seymour [3], testing whether a graph has a $\mathbb{Z}$-linkless embedding or not can be done in polynomial time. Our algorithm partially supplements the result of [4] by providing a $\mathbb{Z}$-linkless embedding.

In the next section we present an algorithm which checks whether a graph has a $\mathbb{Z}$-linkless embedding and computes such an embedding, although this is not yet a polynomial-time algorithm. It amounts to finding an integral vector x such that

$$Mx = L, \tag{1}$$

where M is a matrix with integer coefficients, depending only on the graph, and where L is an integral vector depending on the embedding. That is, we want to find an integral vector y, with $y_1 = 1$, such that $Ny = 0$, where

$$N := \begin{pmatrix} L & M \end{pmatrix}.$$

The number of rows of this matrix can be exponentially large. This forms the main problem in obtaining a polynomial-time algorithm, so we need to reduce the number of rows. The number of columns of this matrix is bounded from above by $|E|^2 + 1$, and so its rank is bounded from above by $|E|^2 + 1$. The main result of this paper is a polynomial-time algorithm to compute a set of rows of N that is of polynomial size and that generates the row space of this matrix for the case that the graph has a $\mathbb{Z}$-linkless embedding (which we can assume by the theorem of Robertson, Seymour, and Thomas). Here it should be stressed that our algorithm to compute a set of rows of N that is of polynomial size and that generates the row space of N need not work for the case that the graph has no $\mathbb{Z}$-linkless embedding. Since finding an integral vector y with $y_1 = 1$ such that $Ny = 0$ can be done in polynomial time if N has polynomial size (see [7]), we can find in polynomial time a vector x such that (1) holds.

To obtain a set of rows of N that generates the row space of M, we will introduce the notion of symmetric 2-cycles. A symmetric 2-cycle on a graph $G = (V, E)$ is a function $d : E \times E \to \mathbb{Z}$ with the following properties: $d(e, f) = d(f, e)$ for all $e, f \in E$, $d(e, f) = 0$ if e and f have a vertex in common, and $d(\cdot, f)$ is a circulation for each $f \in E$; in Section 3 we elaborate on symmetric 2-cycles. The rows of M are in one-to-one correspondence with certain symmetric 2-cycles. There are also special symmetric 2-cycles on subgraphs homeomorphic to K_5 or to $K_{3,3}$; we call these symmetric 2-cycles

Kuratowski 2-cycles. We will show that these two types of symmetric 2-cycles generate the lattice of all symmetric 2-cycles. For symmetric 2-cycles over $\mathbb{Z}_2$ this is also shown in [6]. In this paper we give a different proof. Notice that Claim (3.4.2) of [6] is false, as each set of generators of the lattice of symmetric 2-cycles on $K_{3,4}$ needs two Kuratowski 2-cycles. Corollary 11 of this paper shows a corrected version.

It can be shown that symmetric 2-cycles on a graph G are in one-to-one correspondence with certain homology classes in $H_2(JG;\mathbb{Z})$, where JG is a two-dimensional space which is a deformation retract of the deleted product of G (here we view G as a topological space). See [1,6] for more information about the deleted product of a graph.

2. An algorithm

A *line* in $\mathbb{R}^3$ is a subset homeomorphic to the closed unit interval. A *circle* in $\mathbb{R}^3$ is a subset homeomorphic to the unit circle. A *frame* in $\mathbb{R}^3$ is a pair (U, V), where

(i) $U \subseteq \mathbb{R}^3$ is closed,
(ii) $V \subseteq U$ is finite,
(iii) $U \setminus V$ has only finitely many arc-wise connected component, called *edges*, and
(iv) for each edge, e, either $|\bar{e} \cap V| = 1$ and $\bar{e}$ is a circle, or $|\bar{e} \cap V| = 2$ and $\bar{e}$ is a line with ends the two members of $\bar{e} \cap V$.

We call V the set of vertices of the frame. So a frame is a graph embedded in $\mathbb{R}^3$.

Let $G = (V, E)$ be a graph, whose edges are oriented arbitrarily. An *oriented circuit* of G is a circuit of G with a specified orientation. For any oriented circuit C of G, define the vector $x_C \in \mathbb{Z}^E$ by

$$x_C(e) = \begin{cases} +1 & \text{if } C \text{ traverses } e \text{ in forward direction,} \\ -1 & \text{if } C \text{ traverses } e \text{ in backward direction, and} \\ 0 & \text{if } C \text{ does not traverse } e. \end{cases} \tag{2}$$

Let $\mathcal{D}$ be the set of all unordered pairs of disjoint oriented circuits of G, and let $\mathcal{P}$ be the set of all unordered pairs of edges of G that have no common end. We define the $\mathcal{D} \times \mathcal{P}$ matrix $M = (m_{i,j})$ by

$$m_{\{C,D\},\{e,f\}} = x_C(e)x_D(f) + x_D(e)x_C(f). \tag{3}$$

We denote the $\{e, f\}$th column of M by $M_{\{e,f\}}$.

Let Γ be a frame in $\mathbb{R}^3$ isomorphic to a graph G. If H is a subgraph of G, we denote by $\Gamma(H)$ the frame whose vertex-set and edge-set corresponds to the one of H under the isomorphism. For each pair of disjoint oriented circuits C, D of G, we define $\mathrm{lk}_\Gamma(C, D) = \mathrm{lk}(\Gamma(C), \Gamma(D))$. A frame is $\mathbb{Z}$*-linkless* if $\mathrm{lk}_\Gamma(C, D) = 0$ for each pair of disjoint oriented circuits C, D of G.

For a frame Γ in $\mathbb{R}^3$ isomorphic to G, define the vector $\mathrm{lk}_\Gamma = (l_i) \in \mathbb{Z}^{\mathcal{D}}$ by

$$l_{\{C,D\}} = \mathrm{lk}_\Gamma(C, D). \tag{4}$$

So this vector depends on the particular frame in $\mathbb{R}^3$.

If we change the frame Γ to a frame Γ' such that no edge is pulled through another edge, then $\mathrm{lk}_{\Gamma'} = \mathrm{lk}_\Gamma$. If an edge e is pulled through another edge f, then $\mathrm{lk}_{\Gamma'} = \mathrm{lk}_\Gamma - M_{\{e,f\}}$ or $\mathrm{lk}_{\Gamma'} = \mathrm{lk}_\Gamma + M_{\{e,f\}}$. Hence, if we are given two frames Γ, Γ' in $\mathbb{R}^3$ isomorphic to a graph G, then $\mathrm{lk}_{\Gamma'} - \mathrm{lk}_\Gamma$ belongs to the lattice generated by the columns of M. (A *lattice* is a discrete subgroup of $\mathbb{R}^n$ for some $n \geqslant 0$.) From this observation it follows that:

Proposition 1. *For any frame Γ in $\mathbb{R}^3$, $2\,\mathrm{lk}_\Gamma$ belongs to the lattice generated by the columns of M.*

Proof. Let Γ' be the frame obtained from Γ by reversing the orientation of $\mathbb{R}^3$. Then $\mathrm{lk}_{\Gamma'} = -\,\mathrm{lk}_\Gamma$. Hence $2\,\mathrm{lk}_\Gamma = \mathrm{lk}_\Gamma - \mathrm{lk}_{\Gamma'}$ belongs to the lattice generated by the columns of M. □

On the other hand, the vector lk_Γ does not always belong to the lattice generated by the column vectors of M.

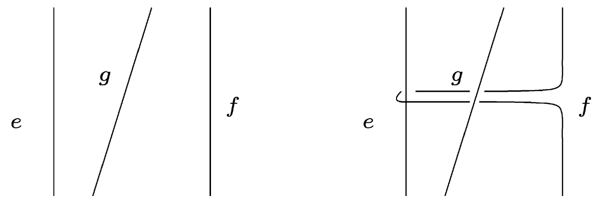

Fig. 2. f pulled through e.

Theorem 2. *Let $G = (V, E)$ be a graph and let Γ be a frame in $\mathbb{R}^3$ isomorphic to G. Then G has a $\mathbb{Z}$-linkless embedding if and only if lk_Γ belongs to the lattice generated by the column vectors of M.*

Proof. If G has a $\mathbb{Z}$-linkless embedding, then there exists a $\mathbb{Z}$-linkless frame Γ' in $\mathbb{R}^3$ isomorphic to G. Since $\mathrm{lk}_{\Gamma'} = 0$, $\mathrm{lk}_\Gamma = \mathrm{lk}_\Gamma - \mathrm{lk}_{\Gamma'}$ belongs to the lattice generated by the column vectors of M.

For the converse we use the following: If Γ_1 is a frame in $\mathbb{R}^3$ isomorphic to G, and m is a column vector of M, then there is a frame Γ_1' in $\mathbb{R}^3$ isomorphic to G such that $\mathrm{lk}_{\Gamma_1'} - \mathrm{lk}_{\Gamma_1} = m$. See Fig. 2 for an illustration with $m = M_{e,f}$ or $m = -M_{e,f}$, depending on how e, f and the pairs of circuits in $\mathcal{D}$ are oriented. So if there exists an $x \in \mathbb{Z}^{\mathcal{P}}$ such $Mx = \mathrm{lk}_\Gamma$, then there is a frame Γ' in $\mathbb{R}^3$ isomorphic to G such that $\mathrm{lk}_{\Gamma'} = 0$. □

From this theorem and the fact that there is an algorithm to solve the equation $Mx = \mathrm{lk}_\Gamma$ for $x \in \mathbb{Z}^{\mathcal{P}}$, it follows that there is an algorithm for finding a $\mathbb{Z}$-linkless embedding of a graph, but it is not polynomial-time, as M has exponentially many rows.

3. Symmetric 2-cycles

Let $G = (V, E)$ be a graph whose edges are oriented arbitrarily. Let $A = (a_{i,j})$ be the $V \times E$ incidence matrix (as the graph G is oriented, A is a matrix with entries in $\{-1, 0, 1\}$). A *circulation* on G is a vector $x : E \to \mathbb{Z}$ satisfying $Ax = 0$. If C is an oriented circuit, then by assigning $+1$ to an edge if it is traversed in forward direction by C, assigning -1 to an edge if it is traversed in backward direction, and assigning 0 to an edge if it is not traversed by C, we obtain a circulation x_C, which we consider as column vector. A *symmetric* 2-*chain* on $G = (V, E)$ is a symmetric $E \times E$ matrix $d : E \times E \to \mathbb{Z}$ such that $d(e, f) = 0$ if e and f have a vertex in common. A *symmetric* 2-*cycle* is a symmetric 2-chain with the additional condition that $Ad = 0$. So for each $e \in E$, $d(e, \cdot)$ is a circulation on G.

Here are some basic examples of symmetric 2-cycles. For each pair of oriented cycles C, D of G, define $d_{C,D} = x_C x_D^T + x_D x_C^T$. If C and D are disjoint oriented circuits of G, then $d_{C,D}$ defines a symmetric 2-cycle.

Choose an edge uv in $K_{3,3}$, and let C_1, D_1 and C_2, D_2 be the two pairs of oriented circuits with $V(C_1) \cap V(D_1) = \{u, v\}$ and $V(C_2) \cap V(D_2) = \{u, v\}$, where we assume that C_1, D_1, C_2, D_2 traverse the edge uv in forward direction. (If we write uv for an edge e, this means that e is oriented from u towards v.) Then $d_{K_{3,3}} = d_{C_1,D_1} - d_{C_2,D_2}$ is a symmetric 2-cycle on $K_{3,3}$.

Choose a vertex v in K_5, let u_1, u_2, u_3, u_4 be the vertices of $K_5 - v$, and let $e_1 = u_1u_2$, $f_1 = u_3u_4$, $e_2 = u_4u_2$, $f_2 = u_1u_3$, $e_3 = u_3u_2$, and $f_3 = u_4u_1$. Let C_i, D_i $(i = 1, 2, 3)$ be distinct pairs of oriented circuits with $V(C_i) \cap V(D_i) = \{v\}$, such that C_i and D_i traverse e_i and f_i, respectively, in forward direction, for $i = 1, 2, 3$. Let $d_{K_5} = d_{C_1,D_1} + d_{C_2,D_2} + d_{C_3,D_3}$. Then d_{K_5} is a symmetric 2-cycle on K_5.

Let G be a graph containing a K-subdivision H, where K is either $K_{3,3}$ or K_5. For each edge e in H, let e' be the corresponding edge of K. For each edge e in G, define

$$\epsilon(H, e) = \begin{cases} +1 & \text{if } e \text{ belongs to } H \text{ and occurs with the same orientation as } e', \\ -1 & \text{if } e \text{ belongs to } H \text{ and occurs with the reversed orientation as } e', \\ 0 & \text{if } e \text{ does not belong to } H. \end{cases} \tag{5}$$

For $e, f \in E$, we define

$$d_H(e,f) = \begin{cases} \epsilon(H,e)\epsilon(H,f)d_K(e',f') & \text{if } e, f \text{ are in } H, \\ 0 & \text{otherwise.} \end{cases} \tag{6}$$

A *pentad* is a subgraph H homeomorphic to K_5, and a *hexad* is a subgraph H homeomorphic to $K_{3,3}$. We call these subgraphs *Kuratowski subgraphs*, and we call the symmetric 2-cycles $-d_H$ and d_H *Kuratowski* 2*-cycles* (on H).

A *separation* of a graph $G = (V, E)$ is a pair (G_1, G_2) of subgraphs of G such that $G_1 \cup G_2 = G$ and $E(G_1) \cap E(G_2) = \emptyset$. The *order* of the separation (G_1, G_2) is $|V(G_1) \cap V(G_2)|$. If (G_1, G_2) is a separation of G of order k, we also call (G_1, G_2) a k-separation of G. If (G_1, G_2) is a separation of G of order $\leqslant k$, we call (G_1, G_2) a $(\leqslant k)$-separation.

We denote by $\tilde{L}_G$ the lattice of all symmetric 2-cycles on G. By L_G we denote the lattice generated by all symmetric 2-cycles on G of the form $d_{C,D}$, with C and D disjoint oriented circuits of G. If (G_1, G_2) is a $(\leqslant 2)$-separation of G, we denote by L_{G_1,G_2} the lattice generated by all symmetric 2-cycles on G of the form $d_{C,D}$, with C an oriented circuit of G_1, D an oriented circuit of G_2, and C and D disjoint.

Lemma 3. *Let (G_1, G_2) be a $(\leqslant 1)$-separation of $G = (V, E)$. Then $\tilde{L}_G = \tilde{L}_{G_1} + \tilde{L}_{G_2} + L_{G_1,G_2}$.*

Proof. The inclusion $\tilde{L}_{G_1} + \tilde{L}_{G_2} + L_{G_1,G_2} \subseteq \tilde{L}_G$ is clear. To see the other inclusion, let $d \in \tilde{L}_G$.

Let $b : E \times E \to \mathbb{Z}$ be the symmetric $E \times E$ matrix with $b(e, f) = d(e, f)$ if $e, f \in E(G_1)$ and $b(e, f) = 0$ otherwise, and let $c : E \times E \to \mathbb{Z}$ be the symmetric $E \times E$ matrix with $c(e, f) = d(e, f)$ if $e, f \in E(G_2)$ and $c(e, f) = 0$ otherwise. Then b and c are symmetric 2-cycles on G. By subtracting b and c from d, we see that we may assume that $d(e, f) = 0$ if $e, f \in E(G_1)$ or $e, f \in E(G_2)$.

Order the edges of G_1 arbitrarily as $e_1, e_2, \dots,$ that starts with edges in $\delta_{G_1}(u)$ for each $u \in V(G_1) \cap V(G_2)$, and the edges of G_2 arbitrarily as $f_1, f_2, \dots$. Choose i, j with $d(e_i, f_j) \neq 0$ and $i + j$ minimal. Let C be a circuit of G_1 in the support of the vector $d(f_j, \cdot)$ that contains e_i. Let D be a circuit of G_2 in the support of the vector $d(e_i, \cdot)$ that contains f_j. Then C does not contain $e_1, \dots, e_{i-1}$ and D does not contain $f_1, \dots, f_{j-1}$. The circuits C and D are disjoint. This is clear if (G_1, G_2) is a 0-separation. If (G_1, G_2) is a 1-separation, then C contains an edge e incident to u, and hence by the ordering chosen, e_i is incident to u. Then D does not traverse u, as it is in the support of the vector $d(e_i, \cdot)$. Now orient C and D in such a way that e_i and f_j are forward arcs of C and D, respectively. So $d_{C,D}(e_i, f_j) = 1$. Replacing d by $d - d(e_i, f_j)d_{C,D}$ makes $d(e_i, f_j) = 0$ and leaves $d(e_k, f_l)$ unchanged for $k + l \leqslant i + j$, $(k, l) \neq (i, j)$. Repeating this until we reach $d = 0$, shows the lemma. □

Lemma 4. *Let (G_1, G_2) be a 2-separation of a 2-connected graph G. For $i = 1, 2$, let P_i be a path in G_i connecting both vertices in $V(G_1) \cap V(G_2)$. Let $H_1 := G_1 \cup P_2$ and $H_2 := G_2 \cup P_1$. Then $\tilde{L}_G = \tilde{L}_{H_1} + \tilde{L}_{H_2} + L_{G_1,G_2}$.*

Proof. Let u_1 and u_2 be the vertices in $V(G_1) \cap V(G_2)$. By reorienting the edges of P_1 and P_2, we may assume that P_1 and P_2 are oriented paths. For each edge e in G_1, let $\phi(e)$ be the net inflow in u_1 of $d(e, \cdot)$ when restricted to $E(G_1)$. For edges e, f in $G_1 \cup P_2$ define

$$d_1(e,f) := \begin{cases} d(e,f) & \text{if } e, f \in E(G_1), \\ 0 & \text{if } e, f \in E(P_2), \\ \phi(e) & \text{if } e \in E(G_1) \text{ and } f \in E(P_2), \\ \phi(f) & \text{if } e \in E(P_2) \text{ and } f \in E(G_1). \end{cases} \tag{7}$$

Then $d_1 \in \tilde{L}_{H_1}$. Similarly, we define $d_2 \in \tilde{L}_{H_2}$.

Let $d_3 = d - d_1 - d_2$. So $d_3(e, f) = 0$ if $e, f \in E(G_1)$ or $e, f \in E(G_2)$. Order the edges of G_1 and G_2 as $e_1, e_2, \dots$ and $f_1, f_2, \dots$, respectively, in such a way that the edges in $\delta_{G_1}(u_1)$ occur first among $e_1, e_2, \dots$, and the edges in $\delta_{G_2}(u_2)$ occur first among $f_1, f_2, \dots$. Choose i, j with $d_3(e_i, f_j) \neq 0$ and $i + j$ minimal. Let C be a circuit in the support of $d_3(\cdot, f_j)$ and containing e_i. Let D be a circuit contained in the support of $d(e_i, \cdot)$ and containing f_j.

Then C and D are circuits in G_1 and G_2, respectively, as $d_3(e,f)=0$ if $e,f\in E(G_1)$ or $e,f\in E(G_2)$. Moreover, C and D are disjoint. For suppose they have a vertex in common, say u_1. So $d_3(e,f_j)\neq 0$ for some $e\in\delta_{G_1}(u_1)$. Then $e_i\in\delta_{G_1}(u_1)$, by the choice of the ordering of the edges $e_1,e_2,\ldots$. But since the support of $d_3(e_i,\cdot)$ contains no edges incident with u_1, we arrive at a contradiction.

Choose the orientations of C and D such that e_i and f_j occur in forward direction. Then replacing d_3 by $d_3-d_3(e_i,f_j)d_{C,D}$ gives a reduction. Repeating this shows that $d_3\in L_{G_1,G_2}$. □

Let $e=v_1v_2$ be an edge of a graph $G=(V,E)$ and let d be a symmetric 2-cycle on G. If $d(f,g)=0$ for each pair of edges f,g with $f\in\delta(v_1)$ and $g\in\delta(v_2)$, we define d/e to be the restriction of d to $E\setminus\{e\}\times E\setminus\{e\}$. Then d/e is a symmetric 2-cycle on G/e. Here G/e denotes the graph obtained from G by contracting e.

Lemma 5. *Let G be a graph and e be an edge of G. Then, for any symmetric 2-cycle d' on G/e, there exists a unique symmetric 2-cycle d on G such that $d/e=d'$. Moreover,*

(i) *if $d'=d_{C,D}$ for disjoint oriented circuits C and D of G/e, then $d=d_{C',D'}$ for disjoint oriented circuits C' and D' of G;*
(ii) *if d' is a Kuratowski 2-cycle on some $K_{3,3}$-subdivision H in G/e, then d is a Kuratowski 2-cycle on some $K_{3,3}$-subdivision H' in G, with $H'/e=H$; and*
(iii) *if d' is a Kuratowski 2-cycle on H for some K_5-subdivision H in G/e, then $d=d_{H'}+\alpha d_{C,D}$ for some $\alpha\in\{0,1\}$, some disjoint oriented circuits C and D of G, and a Kuratowski 2-cycle $d_{H'}$ on some K_5- or $K_{3,3}$-subdivision H' in G, contained in a subgraph H'' of G with $H''/e=H$.*

The proof of this lemma is easy. It follows from (i) of this lemma that if $d/e\in L_{G/e}$, then $d\in L_G$.

Lemma 6. *Let $G=(V,E)$ be a 3-connected graph with $|V|>4$. Then G has an edge e such that G/e is 3-connected.*

A proof of this lemma can be found in [2].

Theorem 7. *Let $G=(V,E)$ be a graph, whose edges are oriented arbitrarily. Then $\tilde{L}_G$ is spanned by the symmetric 2-cycles $d_{C,D}$, with C and D disjoint oriented circuits of G, and the Kuratowski 2-cycles of G.*

Proof. We show this by induction on the number of vertices of G. By Lemmas 3 and 4, we may assume that G is 3-connected. The case where $|V|=4$ is easy.

Let d be a symmetric 2-cycle. By Lemma 6, there exists an edge g of G such that G/g is 3-connected. Let g have ends u_1 and u_2. We show that there exist symmetric 2-cycles d_{C_i,D_i}, $i=1,\ldots,k$, and Kuratowski 2-cycles d_{H_i}, $i=1,\ldots,l$, such that

$$d'=d-\sum_{i=1}^{k}d_{C_i,D_i}-\sum_{i=1}^{l}d_{H_i} \tag{8}$$

satisfies $d'(e,f)=0$ for every $e\in\delta(u_1)$ and $f\in\delta(u_2)$. Once we have shown this, d'/g is a symmetric 2-cycle of G/g. By induction, d'/g has the required form. Lemma 5 then shows that d' belongs to the lattice spanned by all $d_{C,D}$, with C and D disjoint oriented circuits of G, and all Kuratowski 2-cycles. Hence, with (8), we have shown the theorem.

Order the edges in $\delta(u_1)\setminus\{g\}$ as $e_1,\ldots,e_k$ in such a way that we start with the edges that connect u_1 to a neighbor of u_2. Similarly, we order the edges in $\delta(u_2)\setminus\{g\}$ as $f_1,\ldots,f_l$ in such a way that we start with the edges that connect u_2 to a neighbor of u_1. Choose i and j with $d(e_i,f_j)\neq 0$ and $i+j$ minimal. Let e_i have ends u_1 and v_1, and let f_j have ends u_2 and v_2.

Let $e_{i'}=u_1w_1$ be an edge in the support of the vector $d(\cdot,f_j)$ that is unequal to e_i, and let $f_{j'}=u_2w_2$ be an edge in the support of the vector $d(e_i,\cdot)$ that is unequal to f_j. These edges exist since $d(\cdot, f_j)$ and $d(e_i, \cdot)$ are circulations. Since e_i and f_j are non-adjacent, we know that $v_1 \neq v_2$. Similarly, we know that $v_1 \neq w_2$ and $v_2 \neq w_1$. We consider now several cases.

In the first case we assume $w_1 \neq w_2$. First suppose that there exist disjoint circuits C and D such that C contains e_i and $e_{i'}$ and such that D contains f_j and $f_{j'}$. Orient C and D such that e_i and f_j are traversed in forward direction by these circuits. Replacing d by $d - d(e_i, f_j)d_{C,D}$ gives a reduction using $i' > i$, $j' > j$.

Next suppose that such circuits do not exist. Then, since $G - u_1 - u_2$ is 2-connected, it contains two disjoint paths Q_1 and Q_2 connecting $\{v_1, v_2\}$ to $\{w_1, w_2\}$. As there are no disjoint circuits C and D with C containing e_i and $e_{i'}$ and with D containing f_j and $f_{j'}$, Q_1 connects v_1 and w_2, and Q_2 connects v_2 and w_1. Since $G - u_1 - u_2$ is 2-connected, there are disjoint paths R_1 and R_2 connecting Q_1 to Q_2. Again using the fact that there are no disjoint circuits C and D with C containing e_i and $e_{i'}$ and with D containing f_j and $f_{j'}$, we see that there exist a circuit F disjoint from g and disjoint paths P_1, P_2, P_3, P_4, openly disjoint from g and starting at v_1, v_2, w_1, w_2, respectively, and ending on F, in the cyclic order P_1, P_2, P_3, P_4. Then $g, F, e_i, e_{i'}, f_j, f_{j'}$ and P_1, P_2, P_3, P_4 form a subdivision H of $K_{3,3}$. Since e_i and f_j belong to disjoint subdivided edges of $K_{3,3}$, we can choose the Kuratowski 2-cycle d_H on H such that $d_H(e_i, f_j) = 1$. Then replacing d by $d - d(e_i, f_j)d_H$ gives a reduction using $i' > i$, $j' > j$.

In the second case we assume that $w_1 = w_2$. Then, by choice of the orderings of the edges $e_1, e_2, \ldots$ and $f_1, f_2, \ldots$ and by the minimality of $i + j$, v_1 is adjacent to u_2, and v_2 is adjacent to u_1. So each of v_1, v_2 and w_1 $(= w_2)$ is adjacent to u_1 and u_2. By the 2-connectivity of $G - u_1 - u_2$, there exist a circuit F disjoint from g, and disjoint paths P_1, P_2, P_3, disjoint from g and starting at v_1, v_2, w_1, respectively, and ending on F. Then g, F, the edges between $\{v_1, v_2, w_1\}$ and $\{u_1, u_2\}$, and P_1, P_2, P_3 form a subdivision H of K_5 or $K_{3,3}$. Since e_i and f_j belong to disjoint subdivided edges of $K_{3,3}$, we can choose the Kuratowski 2-cycle d_H on H such that $d_H(e_i, f_j) = 1$. Then replacing d by $d - d(e_i, f_j)d_H$ gives a reduction.

Hence we may assume that $d(e, f) = 0$ for each $e \in \delta(u_1)$ and $f \in \delta(u_2)$, which concludes the proof. $\square$

Since planar graphs do not contain K_5- or $K_{3,3}$-subdivisions, we obtain

Proposition 8. *A graph G is planar if and only if $\tilde{L}_G = L_G$.*

4. Kuratowski 2-cycles

Let G be a graph, whose edges are oriented arbitrarily. According to Theorem 7, the lattice $\tilde{L}_G$ is spanned by the symmetric 2-cycles $d_{C,D}$, with C and D disjoint oriented circuits of G, and the Kuratowski 2-cycles of G. In this section, we show that if the connectivity of G is sufficiently high, then $\tilde{L}_G$ is spanned by the symmetric 2-cycles $d_{C,D}$, with C and D disjoint oriented circuits of G, and at most one Kuratowski 2-cycle of G.

A *branch* of a graph G is a path in G whose ends have degree $\geqslant 3$ in G and whose internal vertices all have degree 2 in G. Two branches are *adjacent* if they have a common end.

Let G be a graph. We say that $X \subseteq E$ *meets* a branch L of a Kuratowski subgraph H in G if $X \cap E(L) \neq \emptyset$. Let H_1 and H_2 be Kuratowski subgraphs of G. A $(\leqslant 3)$-separation (A, B) of G *divides* H_1 and H_2 if $E(A)$ meets $\leqslant 3$ branches of H_1 and $E(B)$ meets $\leqslant 3$ branches of H_2, or vice versa.

Label the vertices in the color class of size four of $K_{3,4}$ by v_1, v_2, v_3, v_4. We denote by J_2 the graph obtained from $K_{3,4}$ by adding an edge e_0 between two vertices v_3 and v_4. See Fig. 3.

Two hexads or two pentads H_1 and H_2 of G are 1-*adjacent* if there are branches L_1, L_2 of $H_1 \cup H_2$ such that H_i is obtained from $H_1 \cup H_2$ by deleting the interior of L_i $(i = 1, 2)$. A hexad H_1 and a pentad H_2 of G are 1-*adjacent* if there is a branch L of $H_1 \cup H_2$ such that H_2 is obtained from $H_1 \cup H_2$ by deleting the interior of L. Two hexads H_1 and H_2 are 2-*adjacent* if there is a subgraph J of G isomorphic to a subdivision of the graph J_2, and for $i = 1, 2$, H_i may be obtained from J by deleting the vertex corresponding to v_i and the interiors of the branches incident to v_i, and the interior of the branch corresponding to e_0. Two Kuratowski subgraphs H, H' of G *communicate* if there is a sequence

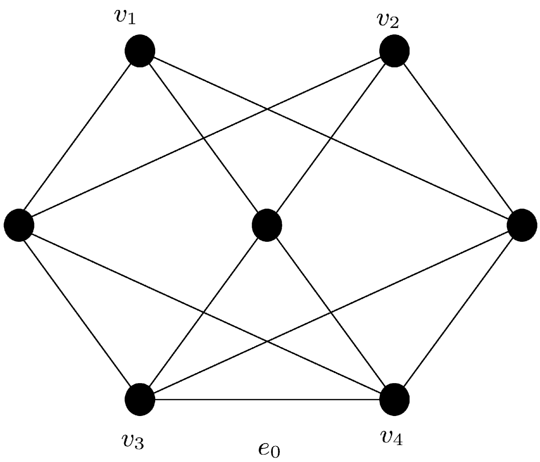

Fig. 3. J_2.

$H = H_1, H_2, \ldots, H_k = H'$ of Kuratowski subgraphs of G such that for $1 \leqslant i \leqslant k-1$, either H_i and H_{i+1} are 1-adjacent, or they are both hexads and are 2-adjacent. A graph G is *Kuratowski connected* if any two Kuratowski subgraphs of G communicate.

Robertson, Seymour, and Thomas [5] showed

Theorem 9. *Let H, H' be Kuratowski subgraphs of G. Then H and H' communicate in G if and only if no $(\leqslant 3)$-separation of G divides H and H'.*

Theorem 10. *Let $G = (V, E)$ be a graph and let H, H' be Kuratowski subgraphs that communicate. If d_H and $d_{H'}$ are Kuratowski 2-cycles on H and H', then $d_H - d_{H'} \in L_G$ or $d_H + d_{H'} \in L_G$.*

Proof. It suffices to show this for the case that either H and H' are 1-adjacent, or that they are both hexads and are 2-adjacent. We may assume that $G = H \cup H'$ and that G is 3-connected.

First we consider the case where H and H' are 1-adjacent. In case H and H' are a hexad and a pentad, we assume that H' is the pentad. There is an edge e of G such that H' is obtained from G by deleting e. Suppose that the ends of e are on adjacent branches of H'. Then there exists an edge f and a circuit C disjoint from the ends of f that both belong to H and to H'. Choose d_H and $d_{H'}$ such that $d_H(f, g) = d_{H'}(f, g)$ for each edge g belonging to C. Let $d = d_H - d_{H'}$. Then $d(f, g) = 0$ for each edge g belonging to C, so the support of $d(f, \cdot)$ is empty. Therefore we may view d as a symmetric 2-cycle of $G \setminus f$. Since $G \setminus f$ is planar, $d \in L_{G\setminus f} \subseteq L_G$. Suppose next that the ends of e are not on adjacent branches of H'. Choose d_H and $d_{H'}$ such that $d_H(f, g) = d_{H'}(f, g)$ for each edge f incident to one end of e and each edge g incident to the other end of e. Let $d = d_H - d_{H'}$. Then $d(f, g) = 0$ for each edge f incident to one end of e and each edge g incident to the other end of e. Then d/e is a symmetric 2-cycle of G/e. Since G/e is planar, d/e belongs to $L_{G/e}$, so $d \in L_G$.

We next consider the case where H and H' are two hexads that are 2-adjacent. We may assume that H and H' are isomorphic to $K_{3,3}$, that H is obtained from J_2 by deleting vertex v_1 and edge e_0, and that H' is obtained from J_2 by deleting v_2 and edge e_0. Choose d_H and $d_{H'}$ such that $d_H(e, f) = d_{H'}(e, f)$ if e is an edge incident to v_3 and f is an edge incident to v_4. Let C_1, C_2, and C_3 be the oriented circuits in J_2 of size three that traverse edge e_0 in forward direction. Let D_1, D_2, and D_3 be oriented circuits such that C_i is vertex-disjoint from D_i for $i = 1, 2, 3$, and, if x_{D_i} is the circulation of D_i $(i = 1, 2, 3)$, then $x_{D_1} + x_{D_2} + x_{D_3} = 0$. If $d_{C_i,D_i} = x_{D_i} x_{C_i}^T + x_{C_i} x_{D_i}^T$ for $i = 1, 2, 3$, then $d_H - d_{H'} = \alpha(d_{C_1,D_1} + d_{C_2,D_2} + d_{C_3,D_3})$, where $\alpha \in \{-1, 1\}$. $\square$

Corollary 11. *Let $G = (V, E)$ be a Kuratowski connected graph. Then $\tilde{L}_G$ can be generated by 2-cycles $d \in L_G$ and one Kuratowski 2-cycle, if any.*

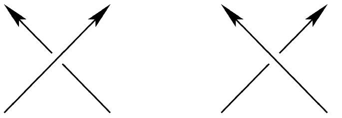

Fig. 4. The sign of a crossing.

5. A characterization of having a $\mathbb{Z}$-linkless embedding

Let $G=(V,E)$ be a graph and consider a frame Γ in $\mathbb{R}^3$ isomorphic to G, and a diagram of Γ into some plane. Let M and lk_Γ be as in (3) and (4). By Theorem 2, G has a $\mathbb{Z}$-linkless embedding in $\mathbb{R}^3$ if and only if there exists a vector $x\in\mathbb{Z}^{\mathcal{P}}$ such that $Mx=\mathrm{lk}_\Gamma$. In this section we apply the following theorem to obtain a characterization for graphs that have a $\mathbb{Z}$-linkless embedding.

Theorem 12. *Let M be an $m\times n$ matrix with entries in $\mathbb{Z}$, and let b be an integral vector in $\mathbb{Z}^m$. Then there exists an integral vector $x\in\mathbb{Z}^n$ such that $Mx=b$ if and only if there is no vector $y\in\mathbb{Q}^m$ such that y^TM is integral while y^Tb is non-integral.*

See Schrijver [7] for a proof of this theorem.

We now first modify the definition of linking number so that it is defined for each symmetric 2-cycles. Let Γ be a frame in $\mathbb{R}^3$ isomorphic to G and consider a diagram of Γ into some plane. The sign of a crossing is defined as in Fig. 4. For $e,f\in E$, define $\mathrm{sign}_\Gamma(e,f)$ as the sum of the signs of all crossings of e with f. For a symmetric 2-cycle d, define the *linking number* of d by

$$\mathrm{link}_\Gamma(d)=\frac{1}{2}\sum_{e,f\in E}\mathrm{sign}_\Gamma(e,f)d(e,f). \tag{9}$$

The linking number is independent of the chosen diagram of the embedding. If C and D are disjoint oriented circuits of G, then $\mathrm{link}_\Gamma(d_{C,D})=2\,\mathrm{lk}_\Gamma(C,D)$. (That $\mathrm{link}_\Gamma(d_{C,D})$ is even for any symmetric 2-cycle of the form $d_{C,D}$ is the reason that we cannot use previous work to find a $\mathbb{Z}_2$-linkless embedding of a graph.) Denote by S_G the linear span over $\mathbb{Q}$ of all symmetric 2-cycles in $\tilde{L}_G$. We extend link_Γ linearly to S_G.

Theorem 13. *Let H be a subdivision of $K_{3,3}$ or K_5, and let Γ be a frame in $\mathbb{R}^3$ isomorphic to H. If d_H is a Kuratowski 2-cycle on H, then $\mathrm{link}_\Gamma(d_H)$ is odd.*

Proof. We give a sketch of the proof. We can find a frame Γ' in $\mathbb{R}^3$ isomorphic to H such that $\mathrm{link}_{\Gamma'}(d_H)$ is odd. Each other frame in $\mathbb{R}^3$ isomorphic to H can be obtained from Γ' by pulling edges through other edges. Hence we may assume that Γ is obtained from Γ' by pulling one edge through another edge once. Since $\mathrm{link}_\Gamma(d_H)-\mathrm{link}_{\Gamma'}(d_H)$ is -2 or 2, we see that $\mathrm{link}_\Gamma(d_H)$ is odd. $\square$

So, if k is odd, then $kd_H\notin L_G$ for each Kuratowski 2-cycle d_H on G. If G has a $\mathbb{Z}$-linkless embedding Γ, then for each integer $k\neq 0$ and each Kuratowski 2-cycle d_H on G, $kd_H\notin L_G$, as $\mathrm{link}_\Gamma(d_{C,D})=0$ for each pair of disjoint oriented circuits C,D, while $\mathrm{link}_\Gamma(d_H)$ is odd. However, if G has no $\mathbb{Z}$-linkless embedding, then there exists a Kuratowski 2-cycle d_H such that $2d_H\in L_G$, which is what we will show now.

Let $G=(V,E)$ be a graph. Recall that $\mathcal{P}$ is the set of all unordered pairs of edges that have no common vertex. We define the function $\phi: S_G\to\mathbb{Z}^{\mathcal{P}}$ by $\phi(d)_{\{e,f\}}=2d(e,f)$ for $\{e,f\}\in\mathcal{P}$.

Lemma 14. *Let G be a Kuratowski connected graph. Then G has no $\mathbb{Z}$-linkless embedding if and only if there exist a Kuratowski 2-cycle d_H, and an integer $k\neq 0$, such that $2kd_H\in L_G$.*

Proof. Let G be a graph that has no $\mathbb{Z}$-linkless embedding. Let Γ be a frame in $\mathbb{R}^3$ isomorphic to G. Let $S = \{d_1, \dots, d_l\}$ be a set of generators of the lattice L_G. Construct the $l \times \mathcal{P}$ matrix

$$M = \begin{pmatrix} \phi(d_1)^T \\ \vdots \\ \phi(d_l)^T \end{pmatrix} \tag{10}$$

and the column vector

$$L = \begin{pmatrix} \mathrm{link}_\Gamma(d_1) \\ \vdots \\ \mathrm{link}_\Gamma(d_l) \end{pmatrix}. \tag{11}$$

Since G has no $\mathbb{Z}$-linkless embedding in $\mathbb{R}^3$, there is no $x \in \mathbb{Z}^{\mathcal{P}}$ such that $Mx = L$. By Theorem 12, there is a vector $z \in \mathbb{Q}^l$ such that $z^T L$ is non-integral and $z^T M$ is integral. Since $z^T L$ is non-integral, $\sum_{i=1}^l z_i \,\mathrm{link}_\Gamma(d_i) = \mathrm{link}_\Gamma(\sum_{i=1}^l z_i d_i)$ is non-integral. Since $z^T M = \sum_{i=1}^l z_i \phi(d_i)^T = \phi(\sum_{i=1}^l z_i d_i)$ is integral, it follows that $d := 2\sum_{i=1}^l z_i d_i$ is a symmetric 2-cycle. Then $\mathrm{link}_\Gamma(d)$ is odd. Since G is Kuratowski connected, there is an integer α and a Kuratowski 2-cycle d_H such that $d - \alpha d_H \in L_G$. Since $\mathrm{link}_\Gamma(d_H)$ is odd, $\mathrm{link}_\Gamma(d_{C,D})$ is even for each pair of disjoint oriented circuits C and D, and $\mathrm{link}_\Gamma(d)$ is odd, we see that α is odd. In particular $\alpha \neq 0$. Let $\beta \neq 0$ be an integer such that $\beta z \in \mathbb{Z}^l$. Then $\beta d = 2\sum_{i=1}^l \beta z_i d_i \in L_G$. Since $\beta d - \alpha\beta d_H \in L_G$, we see that $\alpha\beta d_H \in L_G$. In other words, there is an integer $a \neq 0$ such that $a d_H \in L_G$. Since $\mathrm{link}_\Gamma(d_H)$ is odd, a is an even integer, and so $2kd_H \in L_G$ for an integer $k \neq 0$.

Conversely, let Γ be a $\mathbb{Z}$-linkless frame in $\mathbb{R}^3$ isomorphic to G. Then $\mathrm{lk}_\Gamma(C, D) = 0$ for every pair of disjoint oriented circuits C, D in G. Hence $\mathrm{link}_\Gamma(d_{C,D}) = 0$ for each symmetric 2-cycle $d_{C,D}$ with C and D disjoint oriented circuits of G. So $\mathrm{link}_\Gamma(d) = 0$ for each $d \in L_G$. On the other hand, $\mathrm{link}_\Gamma(d_H)$ is odd for each Kuratowski 2-cycle d_H of G, and in particular, $\mathrm{link}_\Gamma(d_H) \neq 0$. Hence $2kd_H \notin L_G$ for each integer $k \neq 0$. $\square$

The next lemma can also be shown by inspection.

Lemma 15. *For each graph G in the Petersen family, there exists a Kuratowski 2-cycle d_H such that $2d_H \in L_G$.*

Proof. Since G has no $\mathbb{Z}$-linkless embedding and is Kuratowski connected, there exists, by Lemma 14, an integer $k \neq 0$ such that $2kd_H \in L_G$. Take an arbitrary pair of disjoint oriented circuits C, D. We can find a frame Γ in 3-space isomorphic to G such that $\mathrm{link}_\Gamma(d_{C,D}) = 2$ and $\mathrm{link}_\Gamma(d_H) \in \{-1, 1\}$, while $\mathrm{link}_\Gamma(d_{C',D'}) = 0$ for any other pair of disjoint oriented circuits C', D' of G. Hence $k \in \{-1, 1\}$. Replacing d_H by $-d_H$ if necessary, we see that $2d_H \in L_G$. $\square$

Lemma 16. *Let G' be a minor of G. If there is a Kuratowski 2-cycle $d_{H'}$ such that $2d_{H'} \in L_{G'}$, then there is a Kuratowski 2-cycle d_H such that $2d_H \in L_G$.*

Proof. We may assume that G' arises from G by contracting one edge e. We can write $2d_{H'} = \sum_i \alpha_i d_{C'_i,D'_i}$. By Lemma 5, there is a Kuratowski subgraph H of G, a Kuratowski 2-cycle d_H on H, a pair of disjoint oriented circuits C, D of G, and an $\alpha \in \{0, 1\}$, such that $d' = d_H + \alpha d_{C,D}$ satisfies $d'/e = d_{H'}$. By the same lemma there are pairs of disjoint oriented circuits C_i, D_i such that $d_{C_i,D_i}/e = d_{C'_i,D'_i}$. Let $d = 2d_H + 2\alpha d_{C,D} - \sum_i \alpha_i d_{C_i,D_i}$. Then $d(f_1, f_2) = 0$ for each pair of edges f_1, f_2 in $G \setminus e$. This implies that $d(f_1, f_2) = 0$ for each pair of edges f_1, f_2 in G. $\square$

Theorem 17. *The following are equivalent*:

(1) *a graph G has a $\mathbb{Z}$-linkless embedding,*
(2) *for each Kuratowski 2-cycle d_H of G, d_H is not in the linear space spanned by L_G, and*
(3) *$2d_H \notin L_G$ for each Kuratowski 2-cycle d_H of G.*

Proof. (1) ⇒ (2). Suppose that for some Kuratowski 2-cycle d_H of G, d_H is in the linear space spanned by L_G. Then there is a non-zero number k such that $kd_H \in L_G$. So G has no $\mathbb{Z}$-linkless embedding.

(2) ⇒ (3). This is clear.

(3) ⇒ (1). Let G be a graph that has no $\mathbb{Z}$-linkless embedding. Then G has a minor isomorphic to a graph G' in the Petersen family. Hence there is a Kuratowski subgraph H' of G' and a Kuratowski 2-cycle $d_{H'}$ on H' such that $2d_{H'} \in L_{G'}$. By Lemma 16, there is a Kuratowski subgraph H of G and a Kuratowski 2-cycle d_H on H such that $2d_H \in L_G$. $\square$

6. Construction of generators

In this section, we present a polynomial-time algorithm to obtain generators for the lattice L_G of any graph G. The algorithm constructs the generators in a recursive manner: if $e = v_1v_2$ is an edge of a graph G' and $G = G'/e$, then from a set of generators for the lattice L_G, it constructs a set of generators for the lattice $L_{G'}$.

For a graph G' and an edge $e = v_1v_2$ of G', denote by $L_{G',e}$ the lattice of all symmetric 2-cycles $d_{C,D}$, with C and D disjoint oriented circuits such that at least one of them does not contain an end of e. Let $G = G'/e$. Define $\phi : L_{G',e} \to L_G$ by $\phi(d) = d/e$, for $d \in L_{G',e}$. Then ϕ defines an isomorphism between the lattices $L_{G',e}$ and L_G. Therefore, in order to obtain a set of generators for $L_{G'}$ from a set of generators of L_G, we need to add only symmetric 2-cycles $d_{C,D}$, where C and D are disjoint oriented circuits with C containing v_1 and D containing v_2. In case G' has a linkless embedding, we show that for every two edges f_1 and f_2 incident to v_1, and every two edges h_1 and h_2 incident to v_2, there is a collection S of polynomial size of symmetric 2-cycles $d_{C,D}$, with C containing f_1 and f_2, and D containing h_1 and h_2, such that for any pair of oriented circuits D_1, D_2, with D_1 containing f_1 and f_2, and D_2 containing h_1 and h_2, d_{D_1,D_2} is an integral combination of the symmetric 2-cycles in S. In case G' has no linkless embedding, we need to add a Kuratowski 2-cycle to S. There is only a polynomial number of sets of edges $\{f_1, f_2, h_1, h_2\}$, with f_1 and f_2 incident to v_1, and h_1 and h_2 incident to v_2. Hence to obtain $L_{G'}$ from L_G, we need to add to $L_{G',e}$ only a polynomial number of symmetric 2-cycles $d_{C,D}$, where C and D are disjoint oriented circuits with C containing v_1 and D containing v_2.

To construct the set S, we work with G instead of G'. Let v be the vertex arising from contracting e. Let f_1, f_2, h_1, h_2 be distinct edges of G incident to v. Find a pair of oriented circuits C_1, C_2 of G with $V(C_1) \cap V(C_2) = \{v\}$, C_1 containing f_1 and f_2, C_2 containing h_1 and h_2, $\epsilon(C_1, f_1) = +1$, and $\epsilon(C_2, h_1) = +1$. If D_1, D_2 is any pair of oriented circuits of G with $V(D_1) \cap V(D_2) = \{v\}$, D_1 containing f_1 and f_2, D_2 containing h_1 and h_2, $\epsilon(D_1, f_1) = +1$, and $\epsilon(D_2, h_1) = +1$, then $d_{D_1,D_2} - d_{C_1,C_2}$ is a symmetric 2-cycle.

We show how to construct a polynomial number of pairs of oriented circuits C_1^i, C_2^i, $i = 1, \dots, k$, of G, with $V(C_1^i) \cap V(C_2^i) = \{v\}$, C_1^i containing f_1 and f_2, C_2^i containing h_1 and h_2, $\epsilon(C_1^i, f_1) = +1$, and $\epsilon(C_2^i, h_1) = +1$, such that for any pair of oriented circuits D_1, D_2 of G, with $V(D_1) \cap V(D_2) = \{v\}$, D_1 containing f_1 and f_2, D_2 containing h_1 and h_2, $\epsilon(D_1, f_1) = +1$, and $\epsilon(D_2, h_1) = +1$, there are integers $b, c_1, \dots, c_k$, and a Kuratowski 2-cycle d_H if any, such that

$$d_{D_1,D_2} - d_{C_1,C_2} - bd_H - \sum_{i=1}^{k} c_i (d_{C_1^i,C_2^i} - d_{C_1,C_2}) \in L_G. \tag{12}$$

In the remainder of this section we fix G, the edges $f_1 = u_1v$, $f_2 = u_2v$, $h_1 = w_1v$, $h_2 = w_2v$, and the circuits C_1, C_2. Let $P_1 = C_1 - v$ and $P_2 = C_2 - v$. We also fix a pair of oriented circuits D_1, D_2 of G, with $V(D_1) \cap V(D_2) = \{v\}$, D_1 containing f_1 and f_2, D_2 containing h_1 and h_2, $\epsilon(D_1, f_1) = +1$, and $\epsilon(D_2, h_1) = +1$. We may assume that

$$d_{D_1,D_2} - d_{C_1,C_2} \notin L_G, \tag{13}$$

for otherwise (12) follows trivially.

To prove (12), we replace the pair D_1, D_2 by a pair of oriented circuits Z_1, Z_2, with $V(Z_1) \cap V(Z_2) = \{v\}$, Z_1 containing f_1 and f_2, and Z_2 containing h_1 and h_2, such that $d_{D_1,D_2} - d_{Z_1,Z_2} \in L_G$

and the subgraph K of G spanned by $E(C_1) \cup E(C_2) \cup E(Z_1) \cup E(Z_2)$ has a minimal number of edges. From (13), it follows that

$$d_{Z_1,Z_2} - d_{C_1,C_2} \notin L_G. \tag{14}$$

From now on we identify oriented circuits with the circulations they define. That is, if C and D are oriented circuits, and x_C and x_D are the circulations defined by C and D, respectively, then $C - D$ denotes the circulation $x_C - x_D$.

Lemma 18. *Let (A, B) be a $(\leqslant 3)$-separation of K. If $E(A)$ contains edges of at most one of the circuits C_1 and C_2 and of at most one of the circuits Z_1 and Z_2, then A contains no circuit.*

Proof. Suppose for a contradiction that A contains a circuit C. We may assume that $E(A)$ contains edges of only C_1 and Z_1, as the other cases are similar. Then C contains an edge g of Z_1. Orient C so that $\epsilon(C, g) = \epsilon(Z_1, g)$. Let $Z_1' = Z_1 - C$. Then $d_{Z_1,Z_2} - d_{Z_1',Z_2} = d_{C,Z_2} \in L_G$ and $E(C_1) \cup E(C_2) \cup E(Z_1') \cup E(Z_2)$ has fewer edges than K, contradicting the minimality of the number of edges in K. This contradiction shows that A contains no circuit. $\square$

Lemma 19. *Let H be a Kuratowski subgraph of K and let (A, B) be a $(\leqslant 3)$-separation of K with $E(B)$ meeting at most three branches of H. Then $E(A)$ contains at least two edges of f_1, f_2, h_1, h_2.*

Proof. If $v \in V(B) \setminus V(A)$ or $E(A)$ contains at most one edge of f_1, f_2, h_1, h_2, then $E(A)$ contains edges of either C_1 or C_2, and of either Z_1 or Z_2. By Lemma 18, A contains no circuit, hence A is a forest. Hence $E(A)$ meets at most three branches of H. This contradicts that $E(B)$ meets at most three branches of H. $\square$

We call a 3-separation (A, B) of G a *parting* if both $E(A)$ and $E(B)$ contain exactly one edge of f_1, f_2, and exactly one edge of h_1, h_2.

Using Lemma 19, we obtain

Lemma 20. *Let (A, B) be a $(\leqslant 3)$-separation of G dividing two Kuratowski subgraphs of K. Then either (A, B) is a parting, or $f_1, f_2 \in E(A)$ or $h_1, h_2 \in E(A)$.*

We define an ordering $<$ on the collection of all partings by $(A_1, B_1) < (A_2, B_2)$ if $A_1 \subseteq A_2$, $B_2 \subseteq B_1$, and $A_1 \neq A_2$ and $B_1 \neq B_2$. We say that two partings (A_1, B_1) and (A_2, B_2) with $(A_1, B_1) < (A_2, B_2)$ are *adjacent* if there is no parting (A_3, B_3) with $(A_1, B_1) < (A_3, B_3) < (A_2, B_2)$. If (A_1, B_1) and (A_2, B_2) are adjacent partings with $(A_1, B_1) < (A_2, B_2)$, we call the subgraph $K = B_1 \cap A_2 - v$ of G a *patch* if there is a pair of vertex-disjoint paths in K, one connecting the vertex s_1 in $V(P_1) \cap V(A_1 \cap B_1)$ to the vertex t_2 in $V(P_2) \cap V(A_2 \cap B_2)$, and one connecting the vertex s_2 in $V(P_2) \cap V(A_1 \cap B_1)$ to the vertex t_1 in $V(P_1) \cap V(A_2 \cap B_2)$; we call any two of these paths *crossing paths* for K. For each such a patch R, we choose an ordered pair of crossing paths $Q(R)_1, Q(R)_2$, where $Q(R)_1$ connects s_1 to t_2, and $Q(R)_2$ connects s_2 to t_1. We say that a patch T is between patches R and S, if $P_1 \cap T$ lies between $P_1 \cap R$ and $P_1 \cap S$ on P_1 when going from u_1 to u_2. (In that case, $P_2 \cap T$ lies between $P_2 \cap R$ and $P_2 \cap S$ on P_2 when going from w_1 to w_2.)

We call a parting (A, B) of G a *split* if both A and B contain a patch.

We use the next lemma also in Lemma 29.

Lemma 21. *Let H be a Kuratowski subgraph of K. If (A, B) is a parting of G such that $E(B)$ meets at most three branches of H, then $A - v$ contains a patch.*

Proof. Suppose for a contradiction that $A - v$ contains no patch. Then Z_1 has only v in common with $A \cap C_2$ or Z_2 has only v in common with $A \cap C_1$. For $i = 1, 2$, let v_i be the vertex of C_i belonging to $V(A \cap B) \setminus \{v\}$. Then Z_1 does not contain v_2, and Z_2 does not contain v_1. By Lemma 18, Z_1 traverses the path $(A - v) \cap C_1$ and Z_2 traverses the path $(A - v) \cap C_2$. Then $K \cap A$ contains no circuit, contradicting that $E(B)$ meets at most three branches of H. Hence $A - v$ contains a patch. □

From this lemma we immediately obtain

Lemma 22. *A parting of G dividing two Kuratowski subgraphs of K is a split.*

Lemma 22 handles the case where the $(\leqslant 3)$-separation (A, B) of G is a parting. Next we study the case where (A, B) is a $(\leqslant 3)$-separation of G, with $f_1, f_2 \in E(A)$ and $h_1, h_2 \in E(B)$, or vice versa. We show that if (A, B) divides two Kuratowski subgraphs of K, then K is a subdivision of $K_{3,4}$. First we need some lemmas.

Lemma 23. *Let X_1, X_2 and Y_1, Y_2 be pairs of oriented circuits of G, with $V(X_1) \cap V(X_2) = V(Y_1) \cap V(Y_2) = \{v\}$, X_1 and Y_1 containing f_1 and f_2, X_2 and Y_2 containing h_1 and h_2, $\epsilon(X_1, f_1) = \epsilon(Y_1, f_1) = +1$, and $\epsilon(X_2, h_1) = \epsilon(Y_2, h_1) = +1$. Let N be the subgraph of G spanned by $E(X_1) \cup E(X_2) \cup E(Y_1) \cup E(Y_2)$. If there is a $(\leqslant 2)$-separation (A, B) of N with $f_1, f_2 \in E(A)$ and $h_1, h_2 \in E(B)$, then $d_{X_1,X_2} - d_{Y_1,Y_2} \in L_G$.*

Proof. Suppose first that (A, B) is a 1-separation of N. Let $S_1 = Y_1 - X_1$ and let $S_2 = Y_2 - X_2$. Then $d_{Y_1,Y_2} - d_{X_1,X_2} = d_{Y_1,S_2} + d_{S_1,X_2} \in L_G$.

Suppose next that (A, B) is a 2-separation of N. Then either $V(X_1) \cap V(Y_2) = V(A \cap B)$ or $V(Y_1) \cap V(X_2) = V(A \cap B)$. Let $S_1 = Y_1 - X_1$ and let $S_2 = Y_2 - X_2$. If $V(X_1) \cap V(Y_2) = V(A \cap B)$, then $d_{Y_1,Y_2} - d_{X_1,X_2} = d_{Y_1,S_2} + d_{S_1,X_2} \in L_G$. If $V(Y_1) \cap V(X_2) = V(A \cap B)$, then $d_{Y_1,Y_2} - d_{X_1,X_2} = d_{S_1,Y_2} + d_{X_1,S_2} \in L_G$. □

Lemma 24. *Let (A, B) be a $(\leqslant 3)$-separation of K, with $f_1, f_2 \in E(A)$ and $h_1, h_2 \in E(B)$. If A has a circuit containing f_1, f_2, and B has a circuit containing h_1 and h_2, then each of the circuits C_1, C_2, Z_1, Z_2 contains exactly one vertex of $V(A \cap B) \setminus \{v\}$.*

Proof. We first show that each of the circuits C_1, C_2, Z_1, Z_2 contains at most one vertex of $V(A \cap B) \setminus \{v\}$. Suppose for a contradiction that there is one containing both vertices of $V(A \cap B) \setminus \{v\}$, say C_1. Then also Z_2 contains both vertices of $V(A \cap B) \setminus \{v\}$. For otherwise K has a $(\leqslant 2)$-separation (X, Y) with $f_1, f_2 \in E(X)$ and $h_1, h_2 \in E(Y)$. Then $d_{C_1,C_2} - d_{Z_1,Z_2} \in L_G$, by Lemma 23, contradicting (14). Let C be a circuit in A containing f_1, f_2. Orient C such that $\epsilon(C, f_1) = \epsilon(C_1, f_1)$. Then $C_1 - C$ is disjoint from C_2. The subgraph of K spanned by $E(C) \cup E(C_2) \cup E(Z_1) \cup E(Z_2)$ has a 1-separation (X, Y) with $f_1, f_2 \in E(X)$ and $h_1, h_2 \in E(Y)$. By Lemma 23, $d_{C,C_2} - d_{Z_1,Z_2} \in L_G$. Since $d_{C_1,C_2} - d_{Z_1,Z_2} = d_{C,C_2} - d_{Z_1,Z_2} + d_{C_1-C,C_2}$, we see that $d_{C_1,C_2} - d_{Z_1,Z_2} \in L_G$, contradicting (14). The cases where any of the circuits C_2, Z_1, Z_2 contains both vertices of $V(A \cap B) \setminus \{v\}$ can be done similarly.

If any of the circuits C_1, C_2, Z_1, Z_2 contains no vertex of $V(A \cap B) \setminus \{v\}$, then at least one of the circuits contains both vertices of $V(A \cap B) \setminus \{v\}$ or K has a $(\leqslant 2)$-separation (X, Y) with $f_1, f_2 \in E(X)$ and $h_1, h_2 \in E(Y)$. Both cases yield a contradiction. Therefore we can conclude that each of the circuits C_1, C_2, Z_1, Z_2 contains exactly one vertex of $V(A \cap B) \setminus \{v\}$. □

Lemma 25. *If $|V(C_1) \cap V(Z_2)| = 2$ and $|V(C_2) \cap V(Z_1)| = 2$, then K is a subdivision of $K_{3,4}$.*

Proof. Let v_1 and v_2 be the vertices of K such that $\{v, v_1\} = V(C_1) \cap V(Z_2)$ and $\{v, v_2\} = V(C_2) \cap V(Z_1)$.

We show that $C_1 - Z_1$ is a circuit. Suppose for a contradiction that $C_1 - Z_1$ is not a circuit. Then it contains a circuit Z containing at most one of the vertices of $\{v_1, v_2\}$.

If Z contains v_1, let C'_1 be a circuit of $C_1 - Z$ containing f_1 and f_2. Orient C'_1 such that $\epsilon(C'_1, f_1) = +1$. The subgraph of K spanned by $E(C'_1) \cup E(C_2) \cup E(Z_1) \cup E(Z_2)$ has a 2-separation (X, Y) with $f_1, f_2 \in E(X)$ and $h_1, h_2 \in E(Y)$. By Lemma 23, $d_{C'_1,C_2} - d_{Z_1,Z_2} \in L_G$. Since C'_1 does not contain v_1 and v_2, $C_1' - C_1$ is a cycle disjoint from C_2, so $d_{C_1'-C_1,C_2} \in L_G$. Hence $d_{C_1,C_2} - d_{Z_1,Z_2} = d_{C_1',C_2} - d_{Z_1,Z_2} - d_{C_1'-C_1,C_2} \in L_G$, contradicting (14). Similarly, Z does not contain v_2. Thus, Z contains no vertex from $\{v_1, v_2\}$. Let g be an edge of Z with $g \in E(C_1)$. Orient Z so that $\epsilon(Z, g) = \epsilon(C_1, g)$. Let Z_1' be the circuit of $Z_1 - Z$ containing the edges f_1 and f_2. Then $E(C_1) \cup E(C_2) \cup E(Z_1') \cup E(Z_2)$ has fewer edges than $E(C_1) \cup E(C_2) \cup E(Z_1) \cup E(Z_2)$. Since $d_{Z_1,Z_2} - d_{Z_1',Z_2} = d_{Z_1-Z_1',Z_2}$ and $Z_1 - Z_1'$ is a cycle disjoint from Z_2, $d_{Z_1,Z_2} - d_{Z_1',Z_2} = d_{Z_1-Z_1',Z_2} \in L_G$, contradicting the minimality of the number of edges in K.

Thus $C_1 - Z_1$ is a circuit, and, similarly, $C_2 - Z_2$ is a circuit. Hence $C_1 \cap Z_1$ and $C_2 \cap Z_2$ are connected, and are therefore paths. It is now easy to see that K is a $K_{3,4}$-subdivision. $\square$

Lemma 26. *If (A, B) is a $(\leqslant 3)$-separation of K dividing Kuratowski subgraphs, such that $f_1, f_2 \in E(A)$ and $h_1, h_2 \in E(B)$, then K is a subdivision of $K_{3,4}$.*

Proof. Since (A, B) divides Kuratowski subgraphs, both A and B contain a circuit. By Lemma 24, each of the circuits C_1, C_2, Z_1, Z_2 contains exactly one of the vertices in $V(A \cap B) \setminus \{v\}$. By Lemma 25, K is a subdivision of $K_{3,4}$. $\square$

A *flexible split* is a split (A, B) of G such that $V(A \cap B)$ separates in A the two vertices of $\{u_1, u_2, w_1, w_2\} \cap V(A)$, and in B the two vertices of $\{u_1, u_2, w_1, w_2\} \cap V(B)$. Notice that any split of a graph that has an inflexible split is itself also inflexible.

Lemma 27. *If G has a flexible split, then K is a subdivision of $K_{3,4}$.*

Proof. Let (A, B) be a flexible split of G. For $i = 1, 2$, let v_i be the vertex of C_i that belongs to $V(A \cap B) \setminus \{v\}$. There are separations (A_1, A_2) of A and (B_1, B_2) of B such that $V(A_1 \cap A_2) = \{v, v_1, v_2\}$ and $V(B_1 \cap B_2) = \{v, v_1, v_2\}$. If Z_1 contains v_1, then $Z_1 = C_1$, as K has a minimal number of edges. Then also $Z_2 = C_2$. Hence $d_{C_1,C_2} - d_{Z_1,Z_2} \in L_G$, contradicting (14). Therefore Z_1 contains v_2 and Z_2 contains v_1. Since K has a minimal number of edges, K is a subdivision of $K_{3,4}$. $\square$

Lemma 28. *If (A, B) is a $(\leqslant 3)$-separation of G that divides Kuratowski subgraphs of K, then at least one of the following holds*:

(1) *(A, B) is an inflexible split, or*
(2) *K is a subdivision of $K_{3,4}$.*

Proof. By Lemma 20, (A, B) is a parting, or $f_1, f_2 \in E(A)$ or $h_1, h_2 \in E(A)$.

Assume first that (A, B) is a parting. By Lemma 22, (A, B) is a split. If (A, B) is flexible, then, by Lemma 27, K is a subdivision of $K_{3,4}$.

Assume next that $f_1, f_2 \in E(A)$ or $h_1, h_2 \in E(A)$, say $f_1, f_2 \in E(A)$. By Lemma 26, K is a subdivision of $K_{3,4}$. $\square$

For each split (A, B), we choose patches $R \subseteq A$ and $S \subseteq B$ such that between R and S there are no other patches. For the following definitions, we assume that $u_1, w_1 \in V(A)$; the other cases are similar. When traversing P_1 from u_1 to u_2, let x_1 and x_2 be the first and last vertex of $P_1 \cap R$, respectively, and let x_3 and x_4 be the first and last vertex of $P_1 \cap S$, respectively. Similarly, when traversing P_2 from w_1 to w_2, let y_1 and y_2 be the first and last vertex of $P_2 \cap R$, respectively, and let y_3 and y_4 be the first and last vertex of $P_2 \cap S$, respectively. Define $D(A, B)_1$ to be the circuit in G spanned by f_1, $P_1(u_1, x_1)$, $Q(R)_1$, $P_2(y_2, y_3)$, $Q(S)_2$, $P_1(x_4, u_2)$, and f_2. Define $D(A, B)_2$ to be the circuit in G spanned by h_1, $P_2(w_1, y_1)$, $Q(R)_2$, $P_1(x_2, x_3)$, $Q(S)_1$, $P_2(y_4, w_2)$, and h_2. We orient $D(A, B)_1$ and $D(A, B)_2$ such that $\epsilon(D(A, B)_1, f_1) = +1$, and $\epsilon(D(A, B)_2, h_1) = +1$. Notice that $D(A, B)_1, D(A, B)_2$ depend only on the patches R and S separated by the split (A, B).

For a split (A, B) of G, denote by $K(A, B)$ the subgraph of G spanned by $E(C_1) \cup E(C_2) \cup E(D_1(A, B)) \cup E(D_2(A, B))$. By reorienting the edges, we may assume that the edges of $K(A, B)$ are traversed in forward direction by the circuits $C_1, C_2, D(A,B)_1, D(A,B)_2$. We now define a special type of hexads. Denote by $H(A,B)$ the hexad in $K(A,B)$ containing the edges f_1, f_2 and the edge in $E(A) \cap \{h_1, h_2\}$. Notice that $H(A,B)$ is different from $H(B,A)$.

In Lemma 30, we show that, if G has an inflexible split, then each Kuratowski subgraph H of K communicates in G with $H(A,B)$ for a split (A,B) of G. First we show that each $(\leqslant 3)$-separation of G dividing H and $H(A,B)$ is necessarily a split.

Lemma 29. *Let H be a Kuratowski subgraph of K and let (A,B) be an inflexible split of G. If (X,Y) is a $(\leqslant 3)$-separation of G dividing H and $H(A,B)$, then (X,Y) is a split of G, with either $E(X) \cap \{f_1, f_2, h_1, h_2\} = E(A) \cap \{f_1, f_2, h_1, h_2\}$ or $E(Y) \cap \{f_1, f_2, h_1, h_2\} = E(A) \cap \{f_1, f_2, h_1, h_2\}$.*

Proof. By symmetry we may assume that $f_1, h_1 \in E(A)$ and $f_2, h_2 \in E(B)$. Furthermore, we may assume that $E(X)$ meets at most three branches of $H(A,B)$ and that $E(Y)$ meets at most three branches of H.

From Lemma 19, it follows that $v \in V(X)$ and that $E(X)$ contains at least two edges of f_1, f_2, h_1, h_2.

Next we show that $v \in V(X \cap Y)$. For, if $v \in V(X) \setminus V(Y)$, then $E(Y)$ contains edges of either C_1 or C_2, and of either $D_1(A,B)$ or $D_2(A,B)$. So $E(Y)$ meets at most three branches of $H(A,B)$. This contradicts that $E(X)$ meets at most three branches of $H(A,B)$.

It is easily seen that $E(Y)$ contains at least two edges of f_1, f_2, h_1, h_2, so both $E(X)$ and $E(Y)$ contain exactly two edges of f_1, f_2, h_1, h_2.

Next we show that $E(X)$ does not contain h_1 and f_2. For suppose that $h_1, f_2 \in E(X)$. Then $f_1, h_2 \in E(Y)$. $V(X \cap Y)$ separates u_1 from u_2, w_1 from w_2, u_1 from w_1, and u_2 from w_2. Since (A,B) is an inflexible split, the vertices of $V(X \cap Y) \setminus \{v\}$ must be on the branches of $K(A,B)$ containing any of the edges f_1, f_2, h_1, h_2. Since $E(X)$ meets at most three branches of $H(A,B)$, X contains no circuit of $H(A,B)$, so one vertex in $V(X \cap Y) \setminus \{v\}$ is on the branch of $K(A,B)$ containing h_1 and the other vertex is on the branch of $K(A,B)$ containing f_2. Hence there is a 1-separation of (X_1, X_2) of X. Then either $(X_1 \cup Y, X_2)$ or $(X_2 \cup Y, X_1)$ is a 2-separation dividing $H(A,B)$ and H. By Lemma 19, we obtain a contradiction.

In the same manner it can be shown that $E(X)$ does not contain h_2 and f_1.

We next show that $E(X)$ does not contain f_1 and f_2. For suppose that $f_1, f_2 \in E(X)$. Then $h_1, h_2 \in E(Y)$. $V(X \cap Y)$ separates u_1 from w_2, u_1 from w_1, w_1 from u_2, and u_2 from w_2. Since (A,B) is an inflexible split, either one vertex in $V(X \cap Y) \setminus \{v\}$ is on the branch of $K(A,B)$ containing f_1 and the other vertex is on the branch of $K(A,B)$ containing f_2, or one vertex in $V(X \cap Y) \setminus \{v\}$ is on the branch of $K(A,B)$ containing h_1 and the other vertex is on the branch of $K(A,B)$ containing h_2. In both cases, X has a 1-separation (X_1, X_2). Hence there is 2-separation of G dividing $H(A,B)$ and H. By Lemma 19, we obtain a contradiction.

In the same manner it can be shown that X does not contain h_1 and h_2.

Hence $E(X)$ contains either f_1 and h_1, or f_2 and h_2. So (X,Y) is a parting of G with either $E(X) \cap \{f_1, f_2, h_1, h_2\} = E(A) \cap \{f_1, f_2, h_1, h_2\}$ or $E(Y) \cap \{f_1, f_2, h_1, h_2\} = E(A) \cap \{f_1, f_2, h_1, h_2\}$. By Lemma 21, $X - v$ contains a patch. Clearly, $Y - v$ contains a patch. Hence (X,Y) is a split of G. $\square$

Lemma 30. *If G has an inflexible split, then each Kuratowski subgraph H of K communicates with $H(A,B)$ for a split (A,B) of G.*

Proof. Let (A,B) be a split of G such that $E(B)$ meets at most three branches of H, and A is minimal under this property. We assume that $f_1, h_1 \in E(A)$ and $f_2, h_2 \in E(B)$, as the other cases are done similarly. We assert that H and $H(A,B)$ communicate. Suppose for a contradiction that H and $H(A,B)$ do not communicate. Then there is a $(\leqslant 3)$-separation (X,Y) of G such that $E(X)$ meets at most three branches of $H(A,B)$ and $E(Y)$ meets at most three branches of H. By Lemma 29, (X,Y) is a split of G, with either $f_1, h_1 \in E(X)$ or $f_2, h_2 \in E(X)$.

Suppose first that $f_1, h_1 \in E(X)$. By the minimality of A, $A \subseteq X$ and $Y \subseteq B$. Then $E(Y)$ meets at most three branches of $H(A,B)$, contradicting that $E(X)$ meets at most three branches of $H(A,B)$.

Next suppose that $f_2, h_2 \in E(X)$. Then $f_1, h_1 \in E(Y)$. Let R and S be adjacent patches with $R \subseteq A$ and $S \subseteq B$. If Y does not contain R, then $E(Y)$ meets at most two branches of $H(A,B)$, contradicting that $E(X)$ meets at most three branches of $H(A,B)$. If Y contains R, then $A \subseteq Y$. Then $E(A)$ meets at most three branches of H, contradicting that $E(B)$ meets at most three branches of H.

We can conclude that there is no $(\leqslant 3)$-separation (X,Y) dividing H and $H(A,B)$, so H and $H(A,B)$ communicate. $\square$

We denote by $J(A,B)$ the Kuratowski 2-cycle on $H(A,B)$ with $J(A,B)(f,g)=1$ for any edge f in $E(C_1) \cap E(H(A,B))$ and any edge g in $E(C_2) \cap E(H(A,B))$.

Lemma 31. *If G has an inflexible split, then there are integers $a_1,\dots,a_k$ and splits (A_i,B_i), $i=1,\dots,k$, of G, such that*

$$d_{Z_1,Z_2} - d_{C_1,C_2} - \sum_{i=1}^{k} a_i J(A_i,B_i) \in L_G.$$

Proof. By Theorem 7, there exist integers $a_1,\dots,a_k$ and Kuratowski 2-cycles d_{H_i}, $i=1,\dots,k$, of K, such that $d_{Z_1,Z_2} - d_{C_1,C_2} - \sum_{i=1}^{k} a_i d_{H_i} \in L_G$. By Lemma 30, each Kuratowski subgraph H communicates with $H(A,B)$ for a split (A,B). Hence the lemma follows. $\square$

For a split (A,B) of G, define the symmetric 2-cycle $d(A,B)$ of G by

$$d(A,B) = d_{D(A,B)_1, D(A,B)_2} - d_{C_1,C_2}.$$

The next two lemmas show that 'neighboring' hexads of the form $H(A,B)$ communicate or that the difference of their Kuratowski 2-cycles, with the right orientation, is of the form $d(A,B)$.

Lemma 32. *Let G have an inflexible split. If (A_1,B_1) and (A_2,B_2) are splits of G such that $B_1 \cap A_2$ contains a patch, then $H(B_1,A_1)$ and $H(A_2,B_2)$ communicate.*

Proof. This follows because $H(B_1,A_1)$ and $H(A_2,B_2)$ have six branches in common. $\square$

Lemma 33. $J(A,B) + J(B,A) = d(A,B)$.

Proof. Let $d = J(A,B) + J(B,A) - d(A,B)$. Since $d(f_1,g) = d(f_2,g) = d(h_1,g) = d(h_2,g) = 0$ for every edge g in the subgraph $K(A,B)$, we may view d as a symmetric 2-cycle of $K(A,B) - v$. The graph $K(A,B) - v$ is planar and has at most one pair of disjoint oriented circuits C, D, so $d = a d_{C,D}$ for some integer a. Since $J(A,B)(f,g) = 1$ for any edge f in $E(C_1) \cap E(H(A,B))$ and any edge g in $E(C_2) \cap E(H(A,B))$, and $J(B,A)(f,g) = 1$ for any edge f in $E(C_1) \cap E(H(B,A))$ and any edge g in $E(C_2) \cap E(H(B,A))$, we see that $a = 0$. $\square$

Lemma 34. *Let G have an inflexible split and let (A,B) and (A',B') be splits of G. Then there are integers a, $b_1,\dots,b_k$, and splits (A_i,B_i), $i=1,\dots,k$, such that*

$$J(A,B) - a J(A',B') - \sum_{i=1}^{k} b_i d(A_i,B_i) \in L_G.$$

Proof. Apply Lemmas 32 and 33. $\square$

Lemma 35. *If G has an inflexible split, then there are integers b, $a_1,\dots,a_k$, and splits (A_i,B_i), $i=1,\dots,k$, of G, such that*

$$d_{Z_1,Z_2} - d_{C_1,C_2} - \sum_{i=1}^{k} a_i d(A_i,B_i) - b J(A_1,B_1) \in L_G.$$

Proof. The lemma follows from Lemmas 31 and 34. □

Lemma 35 shows (12) for the case where G has an inflexible split. According to Lemma 28, it remains to prove (12) for the case where K is a subdivision of $K_{3,4}$.

Lemma 36. *If K is a subdivision of $K_{3,4}$, then there are integers a, b, a Kuratowski 2-cycle d_H on a Kuratowski subgraph H of G, and a split (A, B) of G such that*

$$d_{Z_1,Z_2} - d_{C_1,C_2} - ad(A, B) - bd_H \in L_G.$$

Proof. Let H_1 and H_2 be hexads of K, with H_1 containing the edges f_1, f_2, h_1, and H_2 containing the edges f_1, f_2, h_2. There are Kuratowski 2-cycle d_{H_1} and d_{H_2} on H_1 and H_2, respectively, such that $d_{Z_1,Z_2} - d_{C_1,C_2} = d_{H_1} + d_{H_2}$.

If H_1 and H_2 communicate in G, then there is an integer b such that $d_{Z_1,Z_2} - d_{C_1,C_2} - bd_{H_1} \in L_G$. Hence we may assume that H_1 and H_2 do not communicate in G. Let (A, B) be a ($\leqslant 3$)-separation of G such that $E(A)$ meets at most three branches of H_2, and $E(B)$ meets at most three branches of H_1. By Lemma 19, $v \in V(A \cap B)$ and $E(A)$ contains exactly two edges of f_1, f_2, h_1, h_2. Since H_1 contains the edges f_1, f_2, h_1, and H_2 contains the edges f_1, f_2, h_2, $E(A)$ does not contain f_1, f_2 or h_1, h_2. Therefore, (A, B) is a parting. By Lemma 22, (A, B) is a split.

Suppose H_1 and $H(A, B)$ do not communicate. Then there is a ($\leqslant 3$)-separation (X, Y) of G such that $E(X)$ meets at most three branches of $H(A, B)$ and $E(Y)$ meets at most three branches of H_1. By Lemma 19, both $E(X)$ and $E(Y)$ contain two edges of f_1, f_2, h_1, h_2, contradicting that both H_1 and $H(A, B)$ contain the edges f_1, f_2, h_1. Similarly, H_2 and $H(B, A)$ communicate. Hence there are integers a_1, a_2 such that $d_{Z_1,Z_2} - d_{C_1,C_2} - a_1 J(A, B) - a_2 J(B, A) \in L_G$. From Lemma 33, it follows that there are integers a, b such that $d_{Z_1,Z_2} - d_{C_1,C_2} - ad(A, B) - bJ(A, B) \in L_G$. □

We now finish the proof of (12).

Theorem 37. *There are integers $b, c_1, \ldots, c_m$, splits (A_i, B_i), $i = 1, \ldots, m$, of G, and a Kuratowski 2-cycle d_H of G, if any, such that*

$$d_{D_1,D_2} - d_{C_1,C_2} - bd_H - \sum_{i=1}^{m} c_i d(A_i, B_i) \in L_G. \tag{15}$$

Proof. By Theorem 7, there are integers $a_1, \ldots, a_m$, Kuratowski subgraphs H_i, $i = 1, \ldots, m$, of K, and Kuratowski 2-cycles d_{H_i}, $i = 1, \ldots, m$, on these Kuratowski subgraphs, such that $d_{Z_1,Z_2} - d_{C_1,C_2} - \sum_{i=1}^{m} a_i d_{H_i} \in L_G$.

If $m \leqslant 1$, then the theorem is clear. Hence we may assume that $m \geqslant 2$. We assume that the Kuratowski subgraphs H_1 and H_2 do not communicate in G. Let (A, B) be a ($\leqslant 3$)-separation of G such that A meets at most three branches of H_1 and B meets at most three branches of H_2. By Lemma 28, (A, B) is an inflexible split or K is a subdivision of $K_{3,4}$.

If (A, B) is an inflexible split, then the theorem follows from Lemma 35. If K is a subdivision of $K_{3,4}$, then the theorem follows from Lemma 36. □

Recall that $G'/e = G$. For $i = 1, 2$, let f_i', h_i', C_i', and $D'(A, B)_i$ be the edges and circuits of G' that correspond in G to f_i, h_i, C_i, and $D(A, B)_i$, respectively. Then

$$(d_{D'(A,B)_1,D'(A,B)_2} - d_{C_1',C_2'})/e = d(A, B).$$

From (15), it follows that for each pair of disjoint oriented circuits D_1', D_2' of G' with D_1' containing f_1' and f_2', D_2' containing h_1' and h_2', $\epsilon(D_1', f_1') = +1$, and $\epsilon(D_2', h_2') = +1$, there are integers $a, b, c_1, \ldots, c_k$, splits (A_i, B_i), $i = 1, \ldots, k$, of G, and a Kuratowski 2-cycle $d_{H'}$ of G' if any, such that

$$d_{D'_1,D'_2} - ad_{C'_1,C'_2} - bd_{H'} - \sum_{i=1}^{k} c_i d_{D'(A,B)_1,D'(A,B)_2} \in L_{G'}.$$

If G' has a $\mathbb{Z}$-linkless embedding, then, by Theorem 17, $kd_{H'} \notin L_{G'}$ for any Kuratowski 2-cycle $d_{H'}$ and every integer $k \neq 0$. Hence, in that case we obtain that

$$d_{D'_1,D'_2} - ad_{C'_1,C'_2} - \sum_{i=1}^{k} b_i d_{D'(A_i,B_i)_1,D'(A_i,B_i)_2} \in L_{G'}. \tag{16}$$

From (16), it follows that, using generators of L_G, generators for the lattice $L_{G'}$ can be found in polynomial time if G' has a $\mathbb{Z}$-linkless embedding. In fact, if we choose the edges e in the right way, the running time to find generators for $L_{G'}$ is $O(n^5)$, where n is the number of vertices in G'. For this we use that graphs that have a linkless embedding have no K_6-minor. Graphs with no K_6-minor contain a vertex of degree at most 7, and so in each graph that has a linkless embedding there exists an edge $e = v_1v_2$ such that the number of distinct edges f_1, f_2, h_1, h_2, with f_1 and f_2 incident to v_1, and h_1 and h_2 incident to v_2 is $O(n^2)$. Since G' has no K_6-minor, it has at most $4n - 10$ edges. Hence we can find in $O(n)$ time a sequence of edges $e_1, e_2, \dots, e_k$, where $k \leqslant n - 1$, such that if v_1 and v_2 are the ends of e_t, the number of set of four distinct edges $\{f_1, f_2, h_1, h_2\}$ in $G'/\{e_1, \dots, e_{t-1}\}$, with f_1 and f_2 incident to v_1, and h_1 and h_2 incident to v_2 is $O(n^2)$. Let v_t denote the vertex in $G'/\{e_1, \dots, e_t\}$ arising from contracting e_t. According to Shiloach [8], for each 4-tuple u_1, u_2, w_1, w_2 of vertices in $G'/\{e_1, \dots, e_t\}$ adjacent to the vertex v_t, finding two vertex-disjoint paths P_1 and P_2 in $G'/\{e_1, \dots, e_t\} - v_t$, with P_1 connecting u_1 and u_2, and P_2 connecting w_1 and w_2, takes $O(n^2)$ time. There are $O(n)$ patches in $G'/\{e_1, \dots, e_t\} - v_t$ and to find crossing paths in each of them can be done in $O(n^2)$ time. For each pair of adjacent patches R, S choose a split (A, B) which separates R and S. In $O(n^2)$ time, we can find the pairs of circuits C_1, C_2 and $D(A, B)_1, D(A, B)_2$, for each split (A, B) we had chosen. Hence the running time to obtain $L_{G'/\{e_1,\dots,e_{t-1}\}}$ from $L_{G'/\{e_1,\dots,e_t\}}$ is $O(n^4)$. Hence in $O(n^5)$ time we can obtain generators for $L_{G'}$.

7. The $\mathbb{Z}$-linkless embedding

Let $G = (V, E)$ be a graph which can be $\mathbb{Z}$-linklessly embedded, and let Γ be a frame. We can check in polynomial time whether Γ is a $\mathbb{Z}$-linkless frame or not as follows. Find a set S of generators for the lattice L_G. This can be done in polynomial time. Check for each generator $d \in S$ if $\mathrm{link}_\Gamma(d) = 0$. If so, Γ is a $\mathbb{Z}$-linkless frame, and if not, then $\mathrm{link}_\Gamma(d) \neq 0$ for some $d \in S$. Let C, D be the pair of disjoint circuits such that $d = d_{C,D}$. Then $\mathrm{lk}_\Gamma(C, D) \neq 0$.

For the construction of a $\mathbb{Z}$-linkless embedding, we use the following theorem.

Theorem 38. *Given a system of rational linear equations, we can decide if it has an integral solution, and if so, find one, in polynomial time.*

See Schrijver [7] for a description of an algorithm.

Theorem 39. *Let $G = (V, E)$ be a graph which can be $\mathbb{Z}$-linklessly embedded. Then a diagram in some plane of a $\mathbb{Z}$-linkless embedding of G can be found in polynomial time.*

Proof. We first construct in polynomial time a diagram Δ of an embedding of G with the property that every pair of edges has a crossing. For this, let $e_1, e_2, \dots, e_m$ be the edges of G, which we orient arbitrarily, and let $v_1, v_2, \dots, v_n$ be the vertices of G. Put $2m$ distinct points $p_1, p_2, \dots, p_m, p_{m+1}, p_{m+2}, \dots, p_{2m}$ on a line l in $\mathbb{R}^2$. The line l divides $\mathbb{R}^2$ into two closed half-plane H_1 and H_2. Connect, for $i = 1, 2, \dots, m$, p_i to p_{m+i} by an arc in H_1. Take a line k in H_2 parallel to l, and map $v_1, v_2, \dots, v_n$ one-to-one on k. If e_h has tail v_i and head v_j, we connect p_h to v_i and p_{m+h} to v_j by arcs in H_2. We label each intersection arbitrarily by undercrossing or overcrossing. It is easily seen that the number of under- and overcrossings in this diagram is $O(n^2)$.

Construct M and L as in Lemma 14, and solve the equation $Mx = L$, $x \in \mathbb{Z}^{\mathcal{P}}$, where $\mathcal{P}$ denotes the set of all unordered pairs of edges that have no vertex in common. By Theorem 38, this can be done in polynomial time. The vector x tells us at which pairs of edges we need to change the embedding in order to make it $\mathbb{Z}$-linkless. Namely, for any pair of edges e, f in G with $x_{\{e,f\}} \neq 0$, choose one of the crossings p of e with f, and locally around p decrease $\text{sign}_{\Gamma}(e, f)$ by $x_{\{e,f\}}$. We can do this so that the increase of the number of crossings is at most $2|x_{\{e,f\}}|$. Hence we can find a diagram Δ' with $\text{sign}_{\Delta'}(e, f) = \text{sign}_{\Delta}(e, f) - x_{\{e,f\}}$ in polynomial time. Then Δ' has the property that $\text{link}_{\Delta'}(C, D) = 0$ for every pair of disjoint oriented circuits C, D of G. $\square$

Acknowledgments

Part of the research was done while I was working at the Mathematical Institute, Freie Universität Berlin and part of it was done while I was visiting the Mathematical Institute, Oxford University. I thank Martin Aigner and Dominic Welsh for their hospitality. I am grateful to Lex Schrijver. Some parts of this paper have improvements due to him. I thank the anonymous referees for checking the paper carefully and for the many suggestions of improvement.

References

[1] A.H. Copeland Jr., C.W. Patty, Homology of deleted products of one-dimensional spaces, Trans. Amer. Math. Soc. 151 (1970) 499–510.
[2] R. Diestel, Graph Theory, second edition, Springer-Verlag, New York, 2000.
[3] N. Robertson, P.D. Seymour, Graph minors. XIII. The disjoint paths problem, J. Combin. Theory Ser. B 63 (1995) 65–110.
[4] N. Robertson, P.D. Seymour, R. Thomas, Sachs' linkless embedding conjecture, J. Combin. Theory Ser. B 64 (1995) 185–227.
[5] N. Robertson, P.D. Seymour, R. Thomas, Kuratowski chains, J. Combin. Theory Ser. B 64 (1995) 127–154.
[6] K.S. Sarkaria, A one-dimensional Whitney trick and Kuratowski's graph planarity criterion, Israel J. Math. 73 (1991) 79–89.
[7] A. Schrijver, Theory of Linear and Integer Programming, Wiley-Intersci. Ser. Discrete Math., John Wiley & Sons, 1990.
[8] Y. Shiloach, A polynomial solution to the undirected two paths problem, J. Assoc. Comput. Mach. 27 (1980) 445–456.